Distribution functions of the sequence phi(n)/n, n in (k,k+N]

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DISTRIBUTION FUNCTIONS OF THE SEQUENCE $\varphi(n)/n$, $n \in (k, k+N]$

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Abstract. It is well known that the sequence $\varphi(n)/n$, $n = 1, 2, \dots$ has a singular asymptotic distribution function. P. Erdős in 1946 found a sufficient condition on the sequence of intervals $(k, k+N]$, such that $\varphi(n)/n$, $n \in (k, k+N]$, has the same singular function. In this note we prove a sufficient and necessary condition. For simplification of necessary condition we express the sum $\sum_{k<n\le k+N}(\omega(n) - \log\log N)^2$, where $\omega(n)$ is the number of different primes divided n.

1. Introduction

Many papers have been devoted to the study of the distribution of the sequence

$$\frac{\varphi(n)}{n}, \quad n = 1, 2, \dots,$$

where φ denotes the Euler phi function. I. J. Schoenberg [S1, S2] established that this sequence has a continuous and strictly increasing asymptotic distribution function (basic properties of distribution functions can be found in [KN, p. 53], [DT, p. 138–157] and [SP, p. 1–7]) and P. Erdős [E1] showed that this function is singular (i.e. has vanishing derivative almost everywhere on $[0, 1]$, see [SP, p. 2–191]). Here the asymptotic distribution function $g_0(x)$ of the sequence $\varphi(n)/n$, $n = 1, 2, \dots$, is defined as

$$g_0(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)}\left(\frac{\varphi(n)}{n}\right), \qquad \text{for every } x \in [0, 1],$$

where $c_{[0,x)}(t)$ is the characteristic function of a subinterval $[0, x)$ of $[0, 1]$. For an interval $(k, k+N]$ and $x \in [0, 1]$ define the step distribution function

$$F_{(k,k+N]}(x) = \frac{1}{N} \sum_{k<n\le k+N} c_{[0,x)}\left(\frac{\varphi(n)}{n}\right).$$

P. Erdős [E2] proved that the limit

$$\lim_{N\to\infty} \frac{\log\log\log k}{N} = 0$$

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implies that

$$F_{(k,k+N]}(x) \to g_0(x), \quad (x \in [0,1]) \tag{1}$$

as $N \to \infty$. In opposite case he found N and k such that $\lim_{N\to\infty} \frac{\log\log\log k}{N} = \frac{1}{2}$ and $F_{(k,k+N]}(x) \not\to g_0(x)$. In this note we give necessary and suffice condition for (1).

2. Necessary and sufficient condition

Theorem 1. *For any two sequences N and k of positive integers we have* (1) *if and only if for every $s = 1, 2, \dots,$*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k<n\le k+N} \sum_{N<d|n} \Phi(d) = 0, \tag{2}$$

where

$$\Phi(d) = \prod_{p|d} \left(\left(1 - \frac{1}{p}\right)^s - 1 \right) \tag{3}$$

for square-free d, $\Phi(d) = 0$ otherwise, and p are primes.

Proof. Applying Weyl's limit relation (see [SP, p. 1–12, Th. 1.8.1.1]) we see that (1) holds if and only if, for every $s = 1, 2, \dots,$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k<n\le k+N} \left(\frac{\varphi(n)}{n}\right)^s = \int_0^1 x^s \mathrm{d}g_0(x), \tag{4}$$

where [P, p. 363]

$$\int_0^1 x^s \mathrm{d}g_0(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\frac{\varphi(n)}{n}\right)^s = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right).$$

We express $\frac{1}{N} \sum_{k<n\le k+N} \left(\frac{\varphi(n)}{n}\right)^s$ by means of

$$\sum_{d|n} \Phi(d) = \left(\frac{\varphi(n)}{n}\right)^s.$$

We have

$$\begin{aligned} \sum_{k<n\le k+N} \sum_{d|n} \Phi(d) &= \sum_{d=1}^{k+N} \Phi(d) \left(\left[\frac{k+N}{d}\right] - \left[\frac{k}{d}\right] \right) \\ &= \sum_{d=1}^{k+N} N \frac{\Phi(d)}{d} + \sum_{d=1}^{k+N} \Phi(d) \left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\} \right), \end{aligned}$$

where $[x]$ is integer part and $\{x\}$ fractional part of x. Since

$$\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\} = \begin{cases} -\left\{\frac{N}{d}\right\} & \text{if } \left\{\frac{k}{d}\right\} + \left\{\frac{N}{d}\right\} < 1 \\ 1 - \left\{\frac{N}{d}\right\} & \text{others} \end{cases} \tag{5}$$

the boundary $k+N$ of summations can be reduced to N and following equality holds

$$\frac{1}{N} \sum_{k<n\le k+N} \left(\frac{\varphi(n)}{n}\right)^s = \sum_{d=1}^{N} \frac{\Phi(d)}{d} + \frac{1}{N} \sum_{d=1}^{N} \Phi(d) \left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right) + \frac{1}{N} \sum_{\substack{N<d\le k+N \\ \left\{\frac{k}{d}\right\}+\frac{N}{d}\ge 1}} \Phi(d). \tag{6}$$

We begin by proving

$$\sum_{\substack{N<d\le k+N \\ \left\{\frac{k}{d}\right\}+\frac{N}{d}\ge 1}} \Phi(d) = \sum_{j=1}^{N} \sum_{\substack{d|k+j \\ d>N}} \Phi(d) \tag{7}$$

for every $k, N = 0, 1, 2, \dots$. Proof: Expressing $k = m_d d + r_d$, where $0 \le r_d < d$, $m_d \ge 0$ are integers, we see that

$$\left\{\frac{k}{d}\right\} + \frac{N}{d} \ge 1 \iff r_d + N \ge d \iff r_d = d - N + i_d$$

where $0 \le i_d < N$. This gives $k = (m_d+1)d - N + i_d$, thus $d(m_d+1) = k + N - i_d$ and, for $d > N$, we have $\left\{\frac{k}{d}\right\} + \frac{N}{d} \ge 1$ if and only if

$$d\left(\left[\frac{k}{d}\right] + 1\right) = k + j \tag{8}$$

for some $j = 1, 2, \dots, N$. In the following we see that for every $d|k+j$ we have that (8) $\iff j \le d$. Really, put $d = k + j - x$ and express $k + j = p_1^{\alpha_1} \dots p_n^{\alpha_n}$, $k + j - x = p_1^{\beta_1} \dots p_n^{\beta_n}$, where p_i are primes. Now d satisfies (8) if and only if

$$(k+j-x)\left(\left[\frac{k}{k+j-x}\right] + 1\right) = k + j \iff \left[\frac{k}{k+j-x}\right] = \frac{x}{k+j-x}$$
$$\iff \left[p_1^{\alpha_1-\beta_1} \dots p_n^{\alpha_n-\beta_n} - \frac{j}{p_1^{\beta_1} \dots p_n^{\beta_n}}\right] = p_1^{\alpha_1-\beta_1} \dots p_n^{\alpha_n-\beta_n} - 1$$
$$\iff j \le p_1^{\beta_1} \dots p_n^{\beta_n}$$

Since $j \le N$ and $d > N$, we have $j < d$ and we conclude that (8) hods and applying it in (6) we get the following basic equality

$$\frac{1}{N} \sum_{k<n\le k+N} \left(\frac{\varphi(n)}{n}\right)^s = \sum_{d=1}^{N} \frac{\Phi(d)}{d} + \frac{1}{N} \sum_{d=1}^{N} \Phi(d) \left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right) + \frac{1}{N} \sum_{k<n\le k+N} \sum_{N<d|n} \Phi(d). \tag{9}$$

Because (see A. G. Postnikov [P, p. 361–363])

$$
\begin{aligned}
&|\Phi(d)| \le \frac{s^{\omega(d)}}{d}, \text{ where } \omega(d) \text{ is the number of different primes which divide } d,\\
&\sum_{d=1}^{N} |\Phi(d)| = O((1+\log N)^s),\\
&\sum_{d=N+1}^{\infty} \frac{|\Phi(d)|}{d} \le \frac{3^s(1+\log N)^s}{N},\\
&\sum_{d=1}^{\infty} \frac{\Phi(d)}{d} = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right),
\end{aligned}
\tag{10}
$$

(9) shows that (1) holds if and only if

$$
\frac{1}{N} \sum_{k<n\le k+N} \sum_{N<d|n} \Phi(d) \to 0
$$

as $N \to \infty$ and the proof of Theorem 1 is complete. $\square$

Notes 1. Using the basic equation (9) and

$$
\frac{1}{N} \sum_{k<n\le k+N} \sum_{N<d|n} \Phi(d) + \frac{1}{N} \sum_{k<n\le k+N} \sum_{d|n, d\le N} \Phi(d) = \frac{1}{N} \sum_{k<n\le k+N} \left(\frac{\varphi(n)}{n}\right)^s
$$

we obtain

$$
\frac{1}{N} \sum_{k<n\le k+N} \sum_{d|n, d\le N} \Phi(d) = \sum_{d=1}^{N} \frac{\Phi(d)}{d} + O\left(\frac{(1+\log N)^s}{N}\right)
$$

and thus, as $N \to \infty$, the left hand side converges to $\prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right)$ uniformly with respect to k.

3. Erdős' approach

For any positive integer n and real $t \ge 2$, denote

$$
n(t) = \prod_{\substack{p|n\\ p\le t}} p, \quad n'(t) = \prod_{\substack{p|n\\ p>t}} p, \text{ and } P(t) = \prod_{p\le t} p,
\tag{11}
$$

where p are primes and the empty product is 1. P. Erdős in [E2] proved (without an explicit error term and for $s = 1$) the following lemma:

Lemma 1. *For every integer k,N and $t=N$ we have*

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s=\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s+O\left(\frac{3^s(1+\log N)^s}{N}\right) \tag{12}$$

for $s=1,2,\dots$

Proof. We have

$$\begin{aligned}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s&=\sum_{k<n\le k+N}\sum_{d|n(t)}\Phi(d)=\sum_{d|P(t)}\Phi(d)\left(\left[\frac{k+N}{d}\right]-\left[\frac{k}{d}\right]\right)\\&=N\sum_{d|P(t)}\frac{\Phi(d)}{d}+\sum_{d|P(t)}\Phi(d)\left(\left\{\frac{k}{d}\right\}-\left\{\frac{k+N}{d}\right\}\right).\end{aligned}$$

Bearing in mind

$$\sum_{d|P(t)}\frac{\Phi(d)}{d}=\prod_{p\le t}\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^s\right),$$

$$\left|\sum_{d|P(t)}\Phi(d)\left(\left\{\frac{k}{d}\right\}-\left\{\frac{k+N}{d}\right\}\right)\right|\le\sum_{d|P(t)}\frac{s^{\omega(d)}}{d}=\prod_{p\le t}\left(1+\frac{s}{p}\right)$$

$$=A(s)(\log t)^s\left(1+O\left(\frac{1}{\log t}\right)\right)\text{ (see [N, p. 110]),}$$

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s=\prod_{p}\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^s\right)+O\left(\frac{3^s(1+\log N)^s}{N}\right)$$

(see [P, p. 363]),

$$\left|1-\prod_{p>N}\left(1-\frac{1}{p}+\frac{1}{p}\left(1-\frac{1}{p}\right)^s\right)\right|\le\sum_{n>N}\frac{|\Phi(n)|}{n}\le\frac{3^s(1+\log N)^s}{N} \tag{13}$$

we obtain (12). □

Next in his method Erdős used implicitly the following theorem:

Theorem 2. *For every two sequences k and N and $t=N$ we have*

$$\left(\prod_{k<n\le k+N}\frac{\varphi(n'(t))}{n'(t)}\right)^{\frac{1}{N}}\to 1\Longrightarrow F_{(k,k+N]}(x)\to g_0(x) \tag{14}$$

for all $x\in[0,1]$.

Proof. Let x_n, $n=1,2,\dots$, be a sequence in the interval $(0,1)$ and define the step distribution function $F_N(x)=\frac{\#\{n\le N;x_n\in[0,x)\}}{N}$. By Riemann-Stiltjes integration for every continuous function $f(x)$ on $[0,1]$ we have $\frac{1}{N}\sum_{n=1}^N f(x_n) = \int_0^1 f(x)\mathrm{d}F_N(x)$. By Helly theorem, if $F_N(x)\to g(x)$, then $\int_0^1 f(x)\mathrm{d}F_N(x) \to \int_0^1 f(x)\mathrm{d}g(x)$.

Now, assume that $\frac{1}{N}\sum_{n=1}^N x_n \to 1$. If $F_N(x)\to g(x)$, then $\int_0^1 x\mathrm{d}g(x) = 1$, which is equivalent to $g(x) = c_0(x)$ (it has step 1 in $x=1$). But $F_N(x)\to c_0(x)$ is equivalent to statistical convergence $x_n\to 1$, i.e. for every $\varepsilon > 0$ and $A_\varepsilon = \{n\le N; x_n \ge 1-\varepsilon\}$ we have $\frac{\# A_\varepsilon}{N}\to 0$. From statistical convergence $x_n\to 1$ follows statistical convergence $f(x_n)\to f(1)$ for every continuous $f(x)$. Furthermore,

$$\frac{1}{N}\sum_{n=1}^N f(x_n) = \int_0^1 F(x)\mathrm{d}F_N(x) \to \int_0^1 f(x)\mathrm{d}c_0(x) = f(1).$$

Thus the limit $\frac{1}{N}\sum_{n=1}^N x_n \to 1$ is equivalent any of the following limits
(i) $\frac{1}{N}\sum_{n=1}^N \log(x_n)\to 0$,
(ii) $\left(\prod_{n=1}^N x_n\right)^{1/N} \to 1$,
(iii) $\frac{1}{N}\sum_{n=1}^N x_n^s \to 1$.
The limits (i)–(iii) also hold assuming restriction $n\in(k,k+N]$.
Put $x_n = \frac{\varphi(n'(t))}{n'(t)}$. Then

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n'(t))}{n'(t)}\right)\to 1 \Longleftrightarrow \frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s \to 1$$

and for $A_\varepsilon \left\{n\in(k,k+N]; \left(\frac{\varphi(n'(t))}{n'(t)}\right)^s < 1-\varepsilon\right\}$ we have $\frac{\# A_\varepsilon}{N}\to 0$ for every $\varepsilon > 0$. Replace in $\frac{\varphi(n)}{n} = \frac{\varphi(n(t))}{n(t)}\frac{\varphi(n'(t))}{n'(t)}$ the $\frac{\varphi(n'(t))}{n'(t)}$ by $1-\varepsilon$ we see

$$\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s \ge \left(\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s\right)(1-\varepsilon) - (1-\varepsilon)\frac{\# A_\varepsilon}{N} \tag{15}$$

and the other hand $\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s \le \frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s$ and Lemma 1 gives

$$(1-\varepsilon)\int_0^1 x^s\mathrm{d}g_0(x) \le \lim_{N\to\infty}\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s \le \int_0^1 x^s\mathrm{d}g_0(x).$$

Thus we have

$$\frac{1}{N}\sum_{k<n\le k+N}\frac{\varphi(n'(t))}{n'(t)} \to 1 \Longrightarrow F_{(k,k+N]}(x)\to g_0(x) \tag{16}$$

and then we use (ii). $\square$

Note 2. In (14) we have only implication, since the right-hand side of (15) has the following precise form

$$(1-\varepsilon)\left(\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s\right)-(1-\varepsilon)\left(\frac{1}{N}\sum_{k<n\le k+N, n\in A_\varepsilon}\left(\frac{\varphi(n(t))}{n(t)}\right)^s\right)$$

and it can be $\left(\frac{1}{N}\sum_{k<n\le k+N,n\in A_\varepsilon}\left(\frac{\varphi(n(t))}{n(t)}\right)^s\right)\to 0$ and $\frac{\#A_\varepsilon}{N}\to\delta>0$.

Finally Erdős prove

Theorem 3. *For every sequence of intervals $(k, k+N]$ we have*

$$\frac{\log\log\log k}{N}\to 0 \Longrightarrow F_{(k,k+N]}(x)\to g_0(x). \tag{17}$$

Proof. For $t=N$, the $n'(t)$, $k<n\le k+N$ are pairwise relatively prime, because the interval $(k, k+N]$ cannot contain two different positive integers divisible by the same prime $p>N$. Denote $M(t)=\prod_{k<n\le k+N} n'(t)$ and $x=\omega(M(t))$. Using the expression

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right) \tag{18}$$

see [MSC, p. 259, VII. 29]) we have

$$\frac{\varphi(M(t))}{M(t)}\ge\prod_{N<p\le x}\left(1-\frac{1}{p}\right)\ge c_1\frac{\log N}{\log x}$$

and (a) $\left(\frac{\log N}{\log x}\right)^{1/N}\to 1$ implies (b) $\left(\frac{\varphi(M(t))}{M(t)}\right)^{1/N}\to 1$. Since

$$e^{c_2x}\le\prod_{N<p\le x}p\le(k+1)(k+2)\dots(k+N)<(k+N)^N$$

we have $c_2x<N\log(k+N)$ and (c) $\left(\frac{\log N}{\log(N\log(k+N))}\right)^{1/N}\to 1$ implies (a) and (b) and finally $F_{(k,k+N]}(x)\to g_0(x)$. But

$$\text{(c)}\Longleftrightarrow\frac{1}{N}\left(\log\frac{\log N}{\log(N\log(k+N))}\right)\to 0\Longleftrightarrow\frac{\log\log\log k}{N}\to 0.$$

□

Note 3. Assume that $P(t)|k$, where $P(t)=\prod_{p\le t}p$ and $t=N$. Analogically as in (11), for a divisor $d|n$ denote $d(t)=\prod_{p|d,p\le t}p$ and $d'(t)=\prod_{p|d,p>t}p$. Since $d(t)|n=k+j$, $d(t)|k$ then $d(t)\le N$ and if we assume $d>N$, then must be $d'(t)>1$. Hence

$$\sum_{N<d|n}\Phi(d)=\sum_{d(t)|n(t)}\Phi(d(t))\sum_{\substack{d'(t)|n'(t)\\ d'(t)>1}}\Phi(d'(t))=\left(\frac{\varphi(n(t))}{n(t)}\right)^s\left(\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s-1\right)$$

which gives

$$\left|\sum_{N<d|n}\Phi(d)\right|\le 1-\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s.$$

Applying Theorem 1 we see that $\lim_{N\to\infty}\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s=1$ implies (1). Since $\frac{\varphi(M(t))}{M(t)}\le\frac{\varphi(n(t))}{n(t)}$ also $\frac{\varphi(M(t))}{M(t)}\to 1$ implies (1). These results directly follows from Erdős Theorem 2 and moreover for general k.

Using Erdős' Lemma 1 we prove the following quantitative form of Theorem 1.

Theorem 4. *For any two sequences N and k of positive integers and for every $s=1,2,\dots,$ we have*

$$\frac{1}{N}\sum_{k<n\le k+N}\sum_{N<d|n}\Phi(d)=\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s-\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s+O\left(\frac{3^s(1+\log N)^s}{N}\right) \tag{19}$$

Proof. Replace n by $n(t)$ in (9). Then

$$\begin{aligned}\frac{1}{N}\sum_{k<n\le k+N}\sum_{d|n(t)}\Phi(d)&=\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s=\\&=\sum_{d=1}^{N}\frac{\Phi(d)}{d}+\frac{1}{N}\sum_{d=1}^{N}\Phi(d)\left(\left\{\frac{k}{d}\right\}-\left\{\frac{k+N}{d}\right\}\right)\\&+\frac{1}{N}\sum_{k<n\le k+N}\sum_{N<d|n(t)}\Phi(d).\end{aligned}$$

Erdős' Lemma 1 implies that for every k,N, $N\to\infty$

$$\frac{1}{N}\sum_{k<n\le k+N}\sum_{N<d|n(t)}\Phi(d)\to 0. \tag{20}$$

Since

$$\sum_{N<d|n}\Phi(d)=\sum_{N<d|n(t)}\Phi(d)+\sum_{\substack{d|n(t)n'(t)\\(d,n'(t))>1}}\Phi(d)$$

and the second sum is equal to $\left(\frac{\varphi(n(t)}{n(t)}\right)^s\left(\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s-1\right)$ we see (19). □

4. Examples

For the optimality of $\log\log\log k$ in Theorem 3, Erdős gave the following example.

Example 1. Divide $P(t)=\prod_{p\le t}p$ into N numbers $A_1,A_2,\dots,A_N$ such that
(i) A_i, $i=1,2,\dots,$ are relatively prime,
(ii) $\frac{\varphi(A_i)}{A_i}<\frac{1}{2}$ for $i=1,2,\dots,$
(iii) if p is the maximal prime in A_i, then for $A_i'=A_i/p$ we have $\frac{\varphi(A_i')}{A_i'}>\frac{1}{2}$.
The part (iii) implies $\frac{\varphi(A_i)}{A_i}>\frac{1}{4}$ and thus

$$\left(\frac{1}{4}\right)^N<\prod_{p\le t}\left(1-\frac{1}{p}\right)=\frac{\varphi(A_1)}{A_1}\dots\frac{\varphi(A_N)}{A_N}<\left(\frac{1}{2}\right)^N.$$

From it, applying (18), we find $N<c_1\log\log t$. By Chinese theorem there exists $k_0<A_1\dots A_N$ such that $k_0\equiv -i \pmod{A)_i}$ for $i=1,2,\dots,N$. Put $k=k_0+A_1\dots A_N$, then we have

$$e^{c_2t}<P(t)=A_1\dots A_N<k$$

which implies $t<c_3\log k$ and $\log\log t<c_4\log\log\log k$. Thus

$$\frac{\log\log\log k}{N}>\frac{1}{c_1c_4}\frac{\log\log t}{\log\log t}.$$

Furthermore for these k and N we have $F_{(k,k+N]}(x)\not\to g_0(x)$ since by (ii)

$$\frac{1}{N}\sum_{k<n\le k+N}\frac{\varphi(n)}{n}<\frac{1}{2}<\frac{1}{N}\sum_{n=1}^{N}\frac{\varphi(n)}{n}=\frac{6}{\pi^2}+O\left(\frac{\log N}{N}\right).$$

In the following Example 2 we find integer sequences k,N, for which (1) holds and $\lim_{N\to\infty}\frac{\log\log\log k}{N}=\infty$.

Example 2. Let $x=x(N)$ be increases arbitrary quickly as $N\to\infty$, e.g. $x(N)=e^{e^{e^N}}$. The left boundary point $k=k(N)$ of the interval $(k,k+N]$ we put as $k=\prod_{p\le x}p$, where p are primes. Let $M^*=\prod_{x<p\le x+y(x)}p$ having the same number of prime divisors as $M(t)$ $(t=N)$. Clearly, to prove $\frac{\varphi(M(t))}{M(t)}\to 1$ it suffices to show

$$\frac{\varphi(M^*)}{M^*}=\prod_{x<p\le x+y(x)}\left(1-\frac{1}{p}\right)\to 1 \tag{21}$$

as $x\to\infty$. Using the expression (18) we have that (21) holds if

$$\frac{\log\left(1+\frac{y(x)}{x}\right)}{\log x}\to 0 \tag{22}$$

as $x\to\infty$. The inequality

$$M^*\le M=\prod_{k<n\le k+N}n'(t)\le(k+N)^N\le(2k)^N$$

leads to $\sum_{x<p\le x+y(x)} \log p \le 2N \sum_{p\le x} \log p$ and thus

$$\sum_{p\le x+y(x)} \log p \le (2N+1) \sum_{p\le x} \log p. \tag{23}$$

Applying the well known inequality (see [L, p. 83])

$$\log 2 \le \liminf_{x\to\infty} \frac{\sum_{p\le x} \log p}{x} \le \limsup_{x\to\infty} \frac{\sum_{p\le x} \log p}{x} \le 2\log 2$$

into (23), then we have, for $x \ge x_0(\varepsilon)$, that $y(x)$ satisfies

$$(\log 2 - \varepsilon)(x + y(x)) \le (2N+1)(2\log 2 + \varepsilon)x$$

which implies $\frac{y(x)}{x} \le cN$, where c is a constant. Thus (21) and consequently (1) holds, if $x = x(N) \ge e^N$. Since $k(N) = \prod_{p\le x(N)} p \ge e^{c_1 x(N)}$, for $x(N) = e^{e^{e^N}}$ we have

$$\lim_{N\to\infty} \frac{\log\log\log k}{N} = \infty,$$

what we asked for.

4. Normal order of $\omega(n)$, $n \in (k, k+N]$

To simplify (2) we study the normal ordering of $\omega(n)$-the umber of distinct prime divisors of n- in the interval $(k, k+N]$ (compare with V.A. Plaksin, see [MSC, p. 156]).

Theorem 4. *For every positive integers k and N we have*

$$\sum_{k<n\le k+N} (\omega(n) - \log\log N)^2 = O(N\log\log N) + \sum_{k<n\le k+N} \sum_{\substack{N<p.q|k+n \\ p\ne q}} 1.$$

A proof follows from the following lemmas.

Lemma 2. *For every positive integers k and N we have*

$$\sum_{k<n\le k+N} \omega(n) = N\log\log N + B.N + O\left(\frac{N}{\log N}\right) + \sum_{k<n\le k+N} \sum_{N<p|n} 1,$$

where the constant $B = \lim_{N\to\infty} \left(\left(\sum_{p\le N} \frac{1}{p} \right) - \log\log N \right)$.

Lemma 3. *For every positive integers k and N we have*

$$\sum_{k<n\le k+N} (\omega(n))^2 = N(\log\log N)^2 + O(N\log\log N) + \sum_{k<n\le k+N}^{N} \sum_{\substack{N<p.q|n\\ p\ne q}} 1 + \sum_{k<n\le k+N} \sum_{N<p|n} 1.$$

Proof of Lemma 1.

$$\sum_{k<n\le k+N} \omega(n) = \sum_{p\le k+N} \left(\left[\frac{k+N}{p}\right] - \left[\frac{k}{p}\right] \right) = \sum_{p\le k+N} \frac{N}{p} + \sum_{p\le k+N} \left\{\frac{k}{p}\right\} - \left\{\frac{k+N}{p}\right\}.$$

The final sum we divide into two parts; $\sum_{p\le N}$ and $\sum_{N<p\le k+N}$. The first sum is $O(\pi(N))$ and for the second we use (5) and the summation method in (7) which gives

$$\sum_{N<p\le k+N} \left\{\frac{k}{p}\right\} - \left\{\frac{k+N}{p}\right\} = \sum_{N<p\le k+N} -\frac{N}{p} + \sum_{k<n\le k+N} \sum_{N<p|n} 1.$$

Thus

$$\sum_{k<n\le k+N} \omega(n) = \sum_{p\le N} \frac{N}{p} + O(\pi(N)) + \sum_{k<n\le k+N} \sum_{N<p|n} 1. \tag{24}$$

Bearing in mind

$$\sum_{p\le N} \frac{1}{p} = \log\log N + B + O\left(\frac{1}{\log N}\right)$$

the proof is finished. □

Notes 2. Similarly to Notes 1 we have

$$\sum_{j=1}^{N} \sum_{\substack{p|k+j\\ p\le N}} 1 = \sum_{p\le N} \frac{N}{p} + O(\pi(N))$$

uniformly on k.

Proof of Lemma 2.

$$\begin{aligned}
\sum_{k<n\le k+N} \omega^2(n) &= \sum_{\substack{\text{all pairs of primes } (p,q)\\ p,q\le k+N}} \sum_{\substack{k<n\le k+N\\ p|n \text{ and } q|n}} 1 \\
&= \sum_{\substack{p.q\le k+N\\ p\ne q}} \left[\frac{k+N}{p.q}\right] - \left[\frac{k}{p.q}\right] + \sum_{p\le k+N} \left[\frac{k+N}{p}\right] - \left[\frac{k}{p}\right] \\
&= \sum_{\substack{p.q\le k+N\\ p\ne q}} \frac{N}{p.q} + \sum_{\substack{p.q\le k+N\\ p\ne q}} \left\{\frac{k}{p.q}\right\} - \left\{\frac{k+N}{p.q}\right\} \\
&\quad + \sum_{p\le k+N} \frac{N}{p} + \sum_{p\le k+N} \left\{\frac{k}{p}\right\} - \left\{\frac{k+N}{p}\right\}
\end{aligned}$$

Since (using (5) and (7))

$$\sum_{\substack{p.q\le N\\ p\neq q}}\left\{\frac{k}{p.q}\right\}-\left\{\frac{k+N}{p.q}\right\}=O(N\log\log N),$$

$$\sum_{\substack{N<p.q\le k+N\\ p\neq q}}\left\{\frac{k}{p.q}\right\}-\left\{\frac{k+N}{p.q}\right\}=\sum_{\substack{N<p.q\le k+N\\ p\neq q}}-\frac{N}{p.q}+\sum_{j=1}^{N}\sum_{\substack{N<p.q|k+j\\ p\neq q}}1,$$

$$\sum_{p\le N}\left\{\frac{k}{p}\right\}-\left\{\frac{k+N}{p}\right\}=O\left(\frac{N}{\log N}\right),$$

$$\sum_{N<p\le k+N}\left\{\frac{k}{p}\right\}-\left\{\frac{k+N}{p}\right\}=\sum_{N<p\le k+N}-\frac{N}{p}+\sum_{j=1}^{N}\sum_{N<p|k+j}1,$$

and moreover

$$\sum_{\substack{p.q\le N\\ p\neq q}}\frac{1}{p.q}=(\log\log N)^2+O(\log\log N),$$

thus the proof of Lemma 2 is finished. □

Finally, the proof of Theorem 4 follows directly from the expression

$$\sum_{k<n\le k+N}(\omega(n)-\log\log N)^2=\sum_{k<n\le k+N}\omega^2(n)-2\log\log N\sum_{k<n\le k+N}\omega(n)+N(\log\log N)^2,$$

from the Lemma 2 and 3 and from

$$\sum_{k<n\le k+N}\sum_{N<p|n}1-2\log\log N\sum_{k<n\le k+N}\sum_{N<p|n}1\le 0.$$

For every integers k, N, the sum $\frac{1}{N}\sum_{k<n\le k+N}\sum_{N<d|n}\Phi(d)$ can be approximate by using the following steps.

a) Let $\omega(n)\le c.\log\log N$ for fixed $c>0$. Since (see [P, p. 361])

$$\left|\sum_{N<d|n}\Phi(d)\right|\le 2^{\omega(n)}\frac{s^{\omega(n)}}{N}$$

and $2^{\omega(n)}\le\left(e^{\log\log N}\right)^{c\log 2}=(\log N)^{c_1}$, $s^{\omega(n)}\le(\log N)^{c_2}$, thus

$$\left|\sum_{N<d|n}\Phi(d)\right|\le\frac{(\log N)^{c_3}}{N}.$$

b) Let $\omega(k+j) > c.\log\log N$. Then $|\omega(k+j) - \log\log N| > (c-1)\log\log N$. Denote by (as in a classical proof of normal order of $\omega(n)$)

$$R_\varepsilon((k,k+N]) = \#\{n \in (k,k+N]; |\omega(n) - \log\log N| > \varepsilon \log\log N\}$$

then

$$R_\varepsilon((k,k+N])\varepsilon^2(\log\log N)^2 \le \sum_{k<n\le k+N} (\omega(n) - \log\log N)^2.$$

Applying Theorem 4 we obtain

$$R_\varepsilon((k,k+N]) \le \frac{1}{\varepsilon^2}\left(O\left(\frac{N}{\log\log N}\right) + \frac{1}{(\log\log N)^2} \sum_{k<n\le k+N} \sum_{\substack{N<p.q|n \\ p\ne q}} 1 \right).$$

Thus we need to estimate the following

$$\frac{R_{c-1}((k,k+N])}{N} \cdot \max_{k<n\le k+N} \left| \sum_{\substack{N<d|n \\ \omega(n)\ge c\log\log N}} \Phi(d) \right|.$$

5. Concluding remarks

1. Lemma 1 implies that every d.f. $g(x)$, $F_{(k,k+N]} \to g(x)$ a.e. on $[0,1]$ must satisfies

$$\int_0^1 x^s \mathrm{d}g(x) \le \int_0^1 x^s \mathrm{d}g_0(x),$$

for every $s = 1, 2, \ldots$.

2. Replaced $1/2$ by $1/N$ in (ii) in Example 1, then by Chinese theorem we can find k, N such that $F_{(k,k+N]}(x) \to c_1(x)$ where d.f. $c_1(x)$ has a step 1 in $x = 1$.

3. A. Schinzel and Y. Wang [SW] proved that for every fixed N the $N-1$-dimensional sequence

$$\left(\frac{\varphi(k+2)}{\varphi(k+1)}, \frac{\varphi(k+3)}{\varphi(k+2)}, \ldots, \frac{\varphi(k+N)}{\varphi(k+N-1)}\right), \quad k = 1, 2, \ldots, N$$

is dense in $[0,\infty)^{N-1}$. Thus, for any given $(\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \in [0,\infty)^{N-1}$ we can select a sequence of k such that

$$\left(\frac{\varphi(k+2)}{\varphi(k+1)}, \frac{\varphi(k+3)}{\varphi(k+2)}, \ldots, \frac{\varphi(k+N)}{\varphi(k+N-1)}\right) \to (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}).$$

Select a subsequence of k such that $\frac{\varphi(k+1)}{k+1} \to \alpha$. Then

$$\left(\frac{\varphi(k+1)}{k+1}, \frac{\varphi(k+2)}{k+2}, \ldots, \frac{\varphi(k+N)}{k+N}\right) \to (\alpha, \alpha\alpha_1, \alpha\alpha_1\alpha_2, \ldots, \alpha\alpha_1\alpha_2\ldots\alpha_{N-1}).$$

Summary, if α_n, $n = 1, 2, \dots$ is an infinite sequence in $[0, \infty)$ then there exists a sequence $k = k(N)$ and $\alpha \in [0, 1]$ such that $F_{(k,k+N]}(x) \to g(x)$ and $g(x)$ is the asymptotic distribution function of the sequence $\alpha\alpha_1 \dots \alpha_n$, $n = 1, 2, \dots$. The constant α is not arbitrary, since

$$\lim_{N\to\infty} \alpha \frac{1}{N} \sum_{n=0}^{N-1} \alpha_1 \dots \alpha_n \le \frac{6}{\pi^2}.$$

4. For completeness we referee some known properties of the d.f. $g_0(x)$. P. Erdoős [E3] estimates modulus of continuity of $g_0(x)$. The explicit construction of $g_0(x)$ is given in H. Davenport [D].

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